QUALITATIVE STUDY OF A CERTAIN EQUATION OF THE THEORY OF PHASE AUTOMATIC FREQUENCY CONTROL

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Methods of bifurcation theory [1] involving the elementary properties of monotonic rotation of the direction field are used for the qualitative investigation of a practically interesting equation. All the possible bifurcations are traced and the domain of existence of a double limit cycle is estimated. The equation in question has been examined by several authors [2-6], but this is the first complete qualitative investigation.

Consider the system

$$d\varphi/dt = y, \qquad dy/dt = \gamma - \sin\varphi - \lambda(1 - d\cos\varphi)y \tag{0.1}$$

We assume that $\gamma \geqslant 0$ and $\lambda \geqslant 0$ (the other possible cases are reducible to this one by substitution of variables).

In the cylindrical phase space (in the strip $-\pi \leqslant \varphi \leqslant \pi$ with identical edges) the equilibrium states are $O_1(\arcsin\gamma, 0)$ (a focus or node), and $O_2(\pi - \arcsin\gamma, 0)$ (a saddle).

The merging and disappearance of singular points is the simplest bifurcation possible in system (0.1). The other possible bifurcations are associated with a change in the stability of the equilibrium state O_1, with bifurcations of saddle-to-saddle separatrices (this is accompanied by the appearance or disappearance of limit cycles), and with the appearance of limit cycles from a trajectory condensation, from the separatrix of a saddle-node point, and from infinity. All of these bifurcations are traceable for system (0.1). Knowledge of all the bifurcations makes it possible to split the parameter space $\gamma > 0$, $\lambda > 0$, d into domains with differing qualitative structures of the decomposition of the phase space into trajectories.

1. Rotation of the field. The parameter plane λ, d can be covered with a net of curves such that variation of the parameters along these curves effects monotonic rotation of the vector field of system (0.1). The difference between the direction fields of system (0.1) with the parameters λ_0 and d_0 and of the altered system with the parameters λ_1 and d_1 for $y \neq 0$ is

$$\lambda_1 - \lambda_0 + (\lambda_0 d_0 - \lambda_1 d_1)\cos\varphi$$

Monotonic rotation is effected if the altered values of the parameters λ_1 and d_1 are chosen in such a way that

$$\lambda_0 d_0 - \lambda_1 d_1 = 0$$

This condition is satisfied if λ and d are varied along the $\mathbf{k}$-curves

$$\lambda d = \mathbf{k}, \qquad \mathbf{k} = \text{const} \; (-\infty < \mathbf{k} < +\infty)$$

The family of $\mathbf{k}$-curves covers the entire surface λd with the exception of the axes λ and d themselves. Contact along the straight line $y = 0$ is false. The curves of the initial and altered axes intersect with tangency along the φ-axis. The difference between the direction fields with variation of the parameter γ is $(\gamma_1 - \gamma_0)/y$. As γ varies, the direction field on the lower and upper half-cylinders rotates in opposite directions.

The straight line $y = 0$ in this case is the contact curve.

2. Sprouting of a cycle out of a focus. The equilibrium state O_1 is a complex focus for the points of the surface

$$\sigma_1 = (P_\varphi' + Q_y')_1 = \lambda (d\sqrt{1-\gamma^2} - 1) = 0$$

On passing through the surface $\sigma_1 = 0$ in the direction of increasing σ_1, the stable focus becomes unstable and sprouts into a single stable limit cycle (the first Liapunov parameter for the points of the surface $\sigma_1 = 0$ has the value $\alpha_3 = -{}^1/_8\pi\lambda(1-\gamma^2)^{-1/4} < 0$).

3. Qualitative structures at the "ends" of k-curves. In order to trace the variation of the qualitative structure of the phase space and the possible bifurcations with monotonic rotation of the field with parameter variation along the **k**-curves we must know the structures of the phase space decomposition at the ends of the **k**-curves for small and large λ (and, correspondingly, for large and small d). Let us represent (0.1) in the form

$$y\,dy + \sin\varphi\,d\varphi = [\gamma - \lambda(1 - d\cos\varphi)\,y]\,d\varphi \qquad (3.1)$$

From (3.1) we infer that

$$\oint_c [\gamma - \lambda(1 - d\cos\varphi)\,y]\,d\varphi = 0$$

if c is limit cycle (3.1). For small γ and λ the limit cycle girding the cylinder is close to one of the curves [7, 1]

$$y_0 = \pm\sqrt{2(\cos\varphi + h)} \qquad (1 < h < \infty)$$

which are the solution of Eq. (3.1) for $\gamma = \lambda = 0$. The value $h = 1$ corresponds to a saddle-to-saddle separatrix. The values of the constant h which isolates the curves of the conservative system near which system (0.1) has limit cycles for small γ and λ on the upper and lower half-cylinders are, respectively, the roots of the equations

$$\psi_1^*(h) = 0, \quad \psi_2^*(h) = 0 \quad (h > 1), \quad \left(\varkappa^2 = \frac{2}{h+1}\right) \qquad (3.2)$$

$$\psi_{1,2}^*(h) = \int_{-\pi}^{\pi} [\gamma - \lambda(1 - d\cos\varphi)\,y_0]\,d\varphi =$$

$$= 2\pi\gamma \mp \lambda\left\{\frac{8E}{\varkappa} - \frac{8d}{3\varkappa^2}[2(\varkappa^2-1)F + (2-\varkappa^2)E]\right\} \equiv \psi_{1,2}(\varkappa)$$

Here F and E are total elliptic integrals of the first and second kind; the upper sign applied to ψ_1, the lower sign to ψ_2. The limit cycle corresponding to the root $\varkappa = \varkappa_0$ is stable if $y_0\psi_{1,2}'(\varkappa_0) > 0$.

The definitions of the functions $\psi_{1,2}$ are supplemented for $\varkappa = 1$ by their limiting values $\psi_1(1) = 2\pi\gamma + {}^8/_3\lambda(d-3)$ and $\psi_2(1) = 2\pi\gamma - {}^8/_3\lambda(d-3)$. From (3.2) we infer that $\psi_1(0) = -\infty, \psi_2(0) = +\infty$ for all d, and also that for $d \geqslant 0$ the derivative $\psi_{1,2}'$ does not change sign in the interval $0 \leqslant \varkappa \leqslant 1$ (Appendix 1). From this we immediately infer that if the condition

$$\psi_1(1) = 2\pi\gamma + {}^8/_3\lambda(d-3) > 0 \qquad (3.3)$$

is satisfied for $d \geqslant 0$ then there is a unique stable limit cycle which lies on the upper half-cylinder, and that if the condition

$$\psi_2(1) = 2\pi\gamma - {}^8/_3\lambda(d-3) < 0 \qquad (3.4)$$

is satisfied, then a unique stable limit cycle lies on the lower half-cylinder. Fulfillment of (3.4) implies fulfillment of (3.3). The requirement that the right side of (3.1) be small $[\gamma < \varepsilon, \lambda < \varepsilon, \lambda|d| < \varepsilon]$ isolates in the plane λd a domain unbounded in d which is adjacent to the axis $\lambda = 0$ and contains the curves $\psi_1(1) = 0$ and $\psi_2(1) = 0$. The equation $\psi_2(1) = 0$ for a small γ in the plane of (nonsmall) parameters λ, d yields, as $\lambda \to 0$, the asymptotic form of the curve which isolates the domain of the parameter plane for whose points in the phase space of system (0.1) there exists a stable limit cycle on both the lower and upper half-cylinders. In this case $d > 0$ and the equilibrium state O_1 is unstable. The qualitative structure of the phase space in this domain appears in Fig. 2 (*1*).

Note. The qualitative picture which appears in Fig. 2 (*1*) is not completely defined by the indicated information, i.e. it is defined to within an even number of limit cycles which possibly surround the equilibrium state. It is not difficult to extend this information by constructing the function ψ_3 and noting (as with ψ_2 and ψ_1) that for small γ and λ a (unique) limit cycle can exist only around a stable focus. This means that the qualitative picture in Fig. 2(*1*) is exact. But the extension of information is pointless in these circumstances, as bifurcation analysis for nonsmall γ and λ does not serve to eliminate the incompleteness anyway.

Let us trace the behavior of the α- and ω-separatrices of the saddle on the upper half-cylinder for large λ, $0 < |d| < 1$ and $0 \leqslant \gamma \leqslant 1$. If the ω-separatrix of the saddle enters the domain above the $y_{\max} = (1+\gamma)/(1-|d|)\lambda$ of the horizontal slope isoclines, then, clearly, limit cycles which gird the cylinder cannot exist. Such parameter values can be chosen for large λ. The directions along which the trajectories of system (0.1) enter the saddle O_2 are defined by the equation

$$\zeta^2 + \lambda(1 + d\sqrt{1-\gamma^2})\zeta - \sqrt{1-\gamma^2} = 0$$

For $0 \leqslant \gamma \leqslant 1$ one root is always negative and corresponds to the direction along which the ω-separatrix enters the saddle. Suppose that the coordinate η_0 of the point of intersection of the straight line with the ω-separatrix of the saddle has been marked on some straight line $\varphi = \varphi_0$. As we move along the $\mathbf{k}$-curves in the parameter space with growing λ, then the vector field rotates monotonically clockwise, and the coordinate η_0 on the straight line $\varphi = \varphi_0$ grows while the maximum of the isocline diminishes. Hence, we can always choose λ and d in such a way that the inequality $(1+\gamma)/\lambda(1-|d|) < \eta_0$ is satisfied.

Limit cycles also cannot exist on the lower half-cylinder for the indicated parameter values, since the existence of a closed contour consisting of the trajectories of system (0.1) on the lower half-cylinder means that

$$\int_{-\pi}^{\pi} [\gamma - \lambda(1 - d\cos\varphi)\,y]\,d\varphi = 0$$

But this is not possible for $y(\varphi) < 0$, $|d| < 1$ and positive λ and γ. For $d < 1$ the equilibrium state O_1 is either an unstable focus or a node. Limit cycles cannot exist around the equilibrium state for $|d| < 1$, since in this case

$$P'_\varphi + Q'_y = -\lambda(1 - d\cos\varphi) \neq 0.$$

The qualitative picture of the phase space for sufficiently large λ on any curve $\lambda d = \mathbf{k}$ is shown in Fig. 2 (*6*).

4. Qualitative pictures of the phase space and possible bifur-

cations for small γ. Let us consider the case $d > 0$. The condition $\psi_2(1) \equiv \equiv 2\pi\gamma - {}^8/_3\lambda(d - 3) = 0$ for small γ and λ yields in the plane λd the asymptotic expression of the curve which isolates the domain of the parameter plane corresponding to the qualitative structure shown in Fig. 2 (*1*). For $\mathbf{k} < {}^3/_4\pi\gamma$ the curves do not enter the domain defined by condition (3.4). For $\mathbf{k} > {}^3/_4\pi\gamma$ there exist $\mathbf{k}$-curves which belong partly to the domain defined by the requirement of smallness of the quantities γ, λ and λd and connecting the domains of the parameter space which correspond to the structures of the phase space decomposition shown in Fig. 2 (*1*, *6*).

By observing the appearance of limit cycles at infinity for small γ and λ, we find that every $\mathbf{k}$-curve which does not belong to the domain $\lambda d < \varepsilon$ also connects the domains of the parameter space corresponding to the phase space structures of Fig. 2 (*1*, *6*) (Appendix 2). Variation of the parameters λ and d along the $\mathbf{k}$-curves effects monotonic rotation of the vector field.

Let us consider the associated behavior of the saddle separatrices. For the structure of the phase decomposition shown in Fig. 2 (*1*) we note the points of intersection of the straight line $\varphi = \varphi_0$ passing through the point O_1 with the α- and ω-separatrices which lie closest to the saddle: P_1 on the ω-separatrix on the lower half-cylinder, P_2 on the α-separatrix on the lower half-cylinder, P_5 on the ω-separatrix on the upper half-cylinder, P_4 on the α-separatrix on the upper half-cylinder, and P_3, which is the second point of intersection on the α-separatrix travelling towards the equilibrium state O_1 (Fig. 1). As λ grows along the $\mathbf{k}$-curves the direction field rotates clockwise and the points P_1 and P_4 rise monotonically, while the points P_2, P_3, P_5 descend monotonically. The possible bifurcations correspond to the merging of the points P_2 and P_1 and (after this has occurred) of the points P_3 and P_1, and also of the points P_4 and P_5. These bifurcations do, in fact, occur, since increases in λ along the $\mathbf{k}$-curves are accompanied by a transition from the structure shown as Fig. 2 (*1*) to the structure shown in Fig. 2 (*6*), and with a change in the signs of the coordinates $y_3 - y_1$ and $y_5 - y_4$ (the subscripts are the same as those of the points).

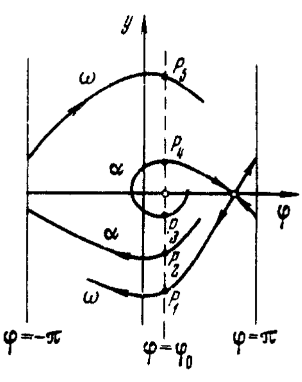

Fig. 1

The sets of points on the plane λd for which the points P_2 and P_1 (a separatrix loop exists below), P_3 and P_1 (a loop exists around the point O_1), or P_4 and P_5 (a loop exists above) coincide, these sets of points form the bifurcation curves $\mathbf{L}^-$, $\mathbf{L}^0$ and $\mathbf{L}^+$ which decompose the plane λd into domains in which the qualitative structures differ in the behavior of the saddle separatrices.

The behavior of the separatrices determines the structure of the decomposition of the phase space into trajectories to within an even number of limit cycles.

The equation $\psi_2(1) = 0$ yields the asymptotic expression for the $\mathbf{L}^+$-curve for small γ and λ; the $\mathbf{L}^-$-curve (that part of it which corresponds to $d \geqslant 0$) begins at the axis $d = 0$ (for $d = 0$ and for small and large λ we have the structure of Fig. 2 (*4*, *6*); as λ increases the field rotates monotonically, so that there exists a unique bifurcation value corresponding to a point of the $\mathbf{L}^+$-curve).

The curve $\mathbf{L}^0$ lies between $\mathbf{L}^-$ and $\mathbf{L}^+$. For small γ and λ the curve $\mathbf{L}^0$ is represented by the equation $d = 3$ (Appendix 3). The curves L^-, L^0 and L^+ intersect each of the k-curves at a single point and then go out to infinity. They cannot intersect, since for $\gamma > 0$ and all λ and d there exists no trajectory decomposition structure for which the saddle separatrices form two loops in the lower and upper half-cylinders (such a structure would be associated with a point at which the three curves L^-, L° and L^+ intersect).

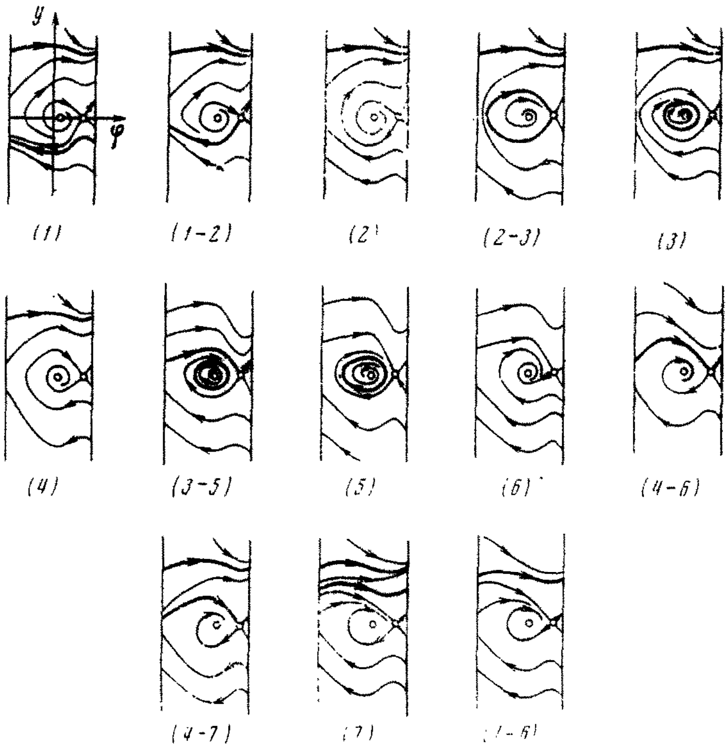

Fig. 2

If we suppose that such a structure does exist for some $\gamma > 0$, then as γ decreases, the monotonousness of rotation of the vector field on the upper and lower half-cylinders (clockwise and counter-clockwise, respectively) results in the destruction of both loops and in the appearance of a structure in which the α-separatrix lies below the ω-separatrix on the lower and upper half-cylinders. Only with such a disposition of the separatrices can there arise a double loop formed by the saddle separatrices with increasing γ. For $\gamma = 0$ and any λ and d such a disposition of the separatrices is impossible because of the symmetry of the direction field with respect to the origin; nor can it arise with increasing γ, since the different directions of the field rotation on the lower and upper half-cylinders with increasing γ the points of the α-separatrices on each half-cylinder can only rise, and the points of the ω-separatrices can only descend.

From what we have said it follows that as λ increases along the k-curves connecting the structures shown in Fig. 2 (*1*, *6*) ($k > {}^3/_4\,\pi\gamma$) there arises a sequence of bifurcations for which the first points to merge are P_2 and P_1, then P_3 and P_1, and finally P_1 and P_5. The curves L^+ and $\sigma_1 = 0$ intersect (this follows from the asymptotic representation of the L^+-curve by the equation $\psi_1(1) = 0$), so that the contraction of the limit cycle to a point with motion along the $\mathbf{k}$-curves (upon crossing of the line $\sigma_1 = 0$) can either precede contraction of the limit cycle to the separatrix loop (upon crossing of the curve L^+) or follow it. The curve L° does not intersect the straight line $\sigma_1 = 0$ on which the stability of the focus changes, on the segment between the d-axis and the L^+-curve (intersection is impossible for $\gamma = 0$ since $P_\varphi' + Q_y'$ does not change sign on $\sigma_1 = 0$, and is therefore impossible for small γ).

The sequence of qualitative structures which arise with the indicated variation of parameters is depicted in Fig. 2 in the form of two possible sequences of coarse structures: (*1—4*, *6*) or (*1—3*, *5*, *6*). The noncoarse structures corresponding to the bifurcation parameter values are denoted by two figures indicating the coarse structures which they separate (see Fig. 2). With motion along the $\mathbf{k}_1$-curves $(0 < \mathbf{k}_1 < {}^3/_4\pi\gamma)$ structure (*1*) drops out of the sequence (the $\mathbf{k}_1$-curves do not intersect the curve $\psi_2(1) = 0$).

Let us turn to the case $d < 0$. Condition (3.3) isolates from the plane λd the domain for whose points the phase space of system (0.1) contains a stable limit cycle on the upper half-cylinder. The equilibrium state O_1 is stable for $d < 0$. The qualitative structure of the phase space in this domain is shown in Fig. 2 (*4*). The curves $\mathbf{k}$ (for $-{}^3/_4\pi\gamma < \mathbf{k} < 0$) connect the domains of the parameter space which correspond to the structures shown in Fig. 2 (*4*,*6*). As λ increases along the $\mathbf{k}$-curves the points P_4 and P_5 on the α- and ω-separatrices of the saddle on the upper half-cylinder (Fig. 1) converge monotonically, merge for some $\lambda = \lambda_0(\mathbf{k})$ (for $d = d_0(\mathbf{k})$, respectively), and then diverge monotonically. The set of points $\lambda_0(\mathbf{k})$, $d_0(\mathbf{k})$ corresponding to the noncoarse bifurcation structure for which the α- and ω-separatrices of the saddle on the upper half-cylinder form a continuous curve which is the extension of the L^+-curve into the domain $d < 0$. One of the $\mathbf{k}$-curves $(-{}^3/_4\pi\gamma < \mathbf{k} < 0)$ passes through every point of L^+.

The saddle parameter $\sigma_2 = (P_\varphi' + Q_y')_2 \equiv -\lambda(1 + d\sqrt{1-\gamma^2})$ changes sign at the straight line $1 + d\sqrt{1-\gamma^2} = 0$ in the parameter plane λ, d. This straight line has just one point of intersection with the L^+-curve (since L^+ cannot have more than one point of intersection with the $\mathbf{k}$-curves).

There exists a unique value $\mathbf{k} = \mathbf{k}_0$ which splits the $\mathbf{k}$-curves into two classes: the $\mathbf{k}_1$-curves $(\mathbf{k}_0 < \mathbf{k}_1 < 0)$ which intersect L^+ for $\sigma_2 < 0$ and the $\mathbf{k}_2$-curves $(-{}^3/_4\pi\gamma < \mathbf{k}_2 < \mathbf{k}_0)$ which intersect L^+ for $\sigma_2 > 0$. For small λ we obtain the structure shown in Fig. 2 (*4*). As λ increases along the $\mathbf{k}_1$-curves the limit cycle descends, and the separatrices on the upper half-cylinder converge.

On the passage through the value of λ corresponding to the intersection of the curves $\mathbf{k}_1$ and L^+ we note the appearance and disintegration of the separatrix loop on the upper half-cylinder to which the stable limit cycle converges (since the saddle parameter $\sigma_2 < 0$). There is no change in the qualitative structure with further variation of the parameters along the $\mathbf{k}_1$-curves. The sequence of qualitative structures as λ increases along the $\mathbf{k}_1$-curves is shown in Fig. 2 (*4*), 2 (*4—6*), 2 (*6*).

As λ increases along the $\mathbf{k}_2$-curves, the limit cycle descends and the separatrices on the upper half-cylinder converge; however, on passage through the value of λ corresponding to the intersection of the curves $\mathbf{k}_2$ and L^+ the disintegration of the separatrix loop is accompanied by the appearance of an unstable limit cycle on the upper half-cylinder (the stable limit cycle cannot converge to a separatrix loop, since the saddle parameter $\sigma_2 > 0$) and we have the structure shown in Fig. 2 (*7*) with two limit cycles girding the upper half-cylinder.

With further increases in the parameter λ along the k_2-curves the limit cycles converge monotonically. Since there are no limit cycles for the structure in Fig. 2 (6), it follows that every k_2-curve has a point with the coordinates $\lambda^{++}(k)$, $d^{++}(k)$ for which the stable and unstable limit cycles merge into a semistable limit cycle. The corresponding noncoarse bifurcation structure is shown in Fig. 2 (7 — 6). The set of points $\lambda^{++}(k)$, $d^{++}(k)$ forms the continuous L^{++}-curve which intersects each of the k_2-curves at a single point to the right of the L^+-curve and begins at the point of intersection of the L^+-curve with the straight line $\sigma_2 = 0$ (Fig. 3). The sequence of qualitative structures as λ increases along the k_2-curves is shown in Fig. 2 (4), 2 (4—7), 2 (7), 2 (7—6) and 2 (6).

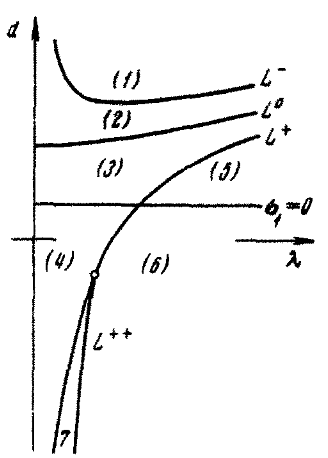

Fig. 3

The curves k_3 ($k_3 < -\,^3/_4\pi\gamma$) do not intersect the L^+-curve (and therefore the L^{++}-curve). Hence, the L^{++}- curve has the same asymptotic expression as the L^+-curve for large $|d|$. On the k_3- curves the structure of the phase space is equivalent to that shown in Fig. 2 (6).

The decomposition of the parameter space λ, d for small γ is shown in Fig. 3; the numbers (1—7) identify the various domains in the parameter space corresponding to the coarse structures in the phase space marked with the same numbers in Fig. 2. The noncoarse structures marked with two numbers in Fig. 2 correspond to the bifurcation curves in Fig. 3 which separate the corresponding domains.

5. The behavior of the bifurcation curves with respect to γ. Other possible bifurcations. Let us trace the changes in the phase space and in the behavior of the bifurcation curves on passage from small positive values of γ to nonsmall values in the range $0 \leqslant \gamma \leqslant 1$. As γ increases the equilibrium states O_1 and O_2 converge. The direction field on the lower and upper half-cylinders rotates monotonically clockwise and counterclockwise, respectively; the stable limit cycles on the upper and lower half-cylinders rise accordingly. If a stable limit cycle on the upper limit cycle exists for some γ_0, then it exists for all $\gamma > \gamma_0$. If for some γ_0 there is a loop on the lower and upper half-cylinders, then as γ increases, the lower loop disintegrates without the appearance of a limit cycle, while the upper loop disintegrates with the appearance of a stable limit cycle. As γ increases the points of the L^--curve separating domains (1 and 2) in Fig. 2 become interior points of domain (2) as γ increases. As γ increases, the points of the L^+-curve become interior points of domains (3 and 4), and the points of the L^{++}-curve become interior points of domains (4 and 7) (or belong to their boundary). The curve L^{++} which begins at the point of intersection of L^+ with the straight line $1 + d\sqrt{1-\gamma^2} = 0$ does not exist above the straight line (the hypothesis of the existence of such points implies the existence of two values γ_1 and γ_0 of the points of intersection of the curves $L^{++}(\gamma_1)$ and $L^{++}(\gamma_0)$, which is impossible because of the monotonicity of rotation of the field on the half-cylinder with monotonic variation of γ), so that the condition $1 + d\sqrt{1-\gamma^2} < 0$ serves as an estimate of the domain of existence of a decomposition structure of the phase cylinder with two limit cycles on the upper half-cylinder, as shown in Fig 2 (7). Domain (7) of the parameter space corresponding to the structure in Fig. 2 (7) descends with increasing γ.

As γ increases to the value $\gamma = 1$ the equilibrium states O_1 and O_2 merge to form a special complex saddle-node point, while domains (1—3, 5 and 7) in Fig. 3 go out to infinity. The only bifurcation curve on the plane λd is the L^+-curve (its existence follows from considerations similar to those adduced in the case of small γ and based on the existence for $\gamma = 1$ of a certain neighborhood of the d-axis for whose points a stable limit cycle exists on the upper half-cylinder). The parameter space and the decomposition structures of the phase space are shown in Fig. 4.

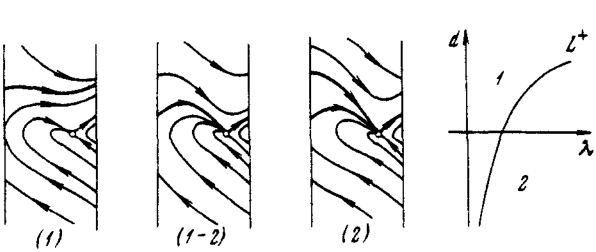

Fig. 4

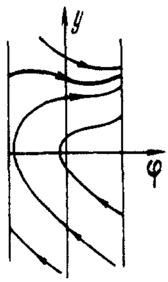

Fig. 5

For $\gamma > 1$ there exists a single structure of the decomposition of the phase space into trajectories. All of the trajectories wind around a stable limit cycle on the upper half-cyclinder (Fig. 5). As γ increases from the value $\gamma = 1$ for λ and d taken from domain (1) in Fig. 4 the saddle-node equilibrium state vanishes. For values of λ and d taken from domain (2) a stable limit cycle arises out of the α-separatrix of the saddle-node.

For $\gamma = 0$ the phase space is symmetric with respect to the origin. The equilibrium states are O_1 (0, 0) and O_2 (0, $\pm \pi$). The existence of a separatrix loop on the upper half-cylinder implies the existence of such a loop on the lower half-cylinder. Such a double loop also forms a closed contour about the equilibrium state O_1. This means that the curves L^-, L° and L^+ coinside. As $\gamma \to 0$ the curves L^- and L^+ converge and merge with the d-axis and the L°-curve for $\gamma = 0$. The points of the L^{++}-curve become interior points of the domain (6) as γ decreases. Domain (7) cannot be preserved for $\gamma = 0$, since this would mean that four cycles would exist even for sufficiently small γ. As $\gamma \to 0$ the L^{++}-curve is "engulfed" by the half-line $\lambda = 0$, $d < -1$. For $\gamma = 0$ the parameter plane λ, d has a single bifurcational L-curve which arises from the merging of the curves L^-, L° and L^+. The curve L begins at the point $\lambda = 0$, $d = 3$ and goes out to infinity. It can intersect neither the straight line $d = 1$ (since $P_\varphi' + Q_y' = -\lambda(1 - d\cos\varphi)$ does not change sign for $|d| \leqslant 1$) nor the straight line $\lambda = 0$ (since it cannot intersect the k-curves at more than one point). On passage through an L-curve along the k-curves with increasing λ and with the appearance of a double loop, the stable limit cycle contracts to each half-loop (since the saddle parameter $(P_\varphi' + Q_y')_2 = -\lambda(1 + d)$ is negative, and since the vector field rotates clockwise). With further changes in λ and disintegration of the loop, the double loop considered as a closed contour surrounding the equilibrium state O_1 gives rise to a stable limit cycle which surrounds this equilibrium state. The limit cycle contracts to a point for $d = 1$ and the focus becomes stable. The parameter space and the phase space decomposition structures for $\gamma = 0$ are shown in Fig. 6.

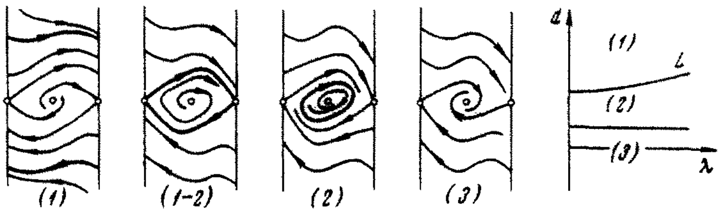

Fig. 6

Appendix 1. a) Making use of the expansions

$$F = \frac{\pi}{2}\left(1 + \frac{\varkappa^2}{4} + \frac{9}{4.16}\varkappa^4 + \ldots\right), \qquad E = \frac{\pi}{2}\left(1 - \frac{\varkappa^2}{4} - \frac{3}{4.16}\varkappa^4 - \ldots\right)$$

we find that the following relation hold for small $\varkappa$:

$$\psi_{1,2}(\varkappa) = 2\pi\gamma \mp \lambda\left(\frac{4\pi}{\varkappa} - \frac{\pi d}{2}\varkappa + \ldots\right) \approx \mp \frac{4\pi\lambda}{\varkappa}$$

so that $\psi_1(0) = -\infty$, $\psi_2(0) = +\infty$.

b) From (3.2) we obtain

$$\psi'_{1,2}(\varkappa) = \mp 8\lambda\left[-\frac{F}{\varkappa^3} - \frac{d}{\varkappa^4}\Phi(\varkappa)\right], \quad \Phi(\varkappa) = (2-\varkappa^2)F - 2E$$

The expression $\Phi(\varkappa)$ is always positive for $\varkappa \neq 0$, so that for $d > 0$ we have

$$\operatorname{sgn}\psi'_{1,2}(\varkappa) = \operatorname{sgn}(\pm\lambda)$$

In fact,

$$\Phi'(\varkappa) = \frac{\varkappa}{1-\varkappa^2}\left[E - (1-\varkappa^2)F\right] \equiv \frac{\varkappa}{1-\varkappa^2}\Phi_1(\varkappa), \qquad \Phi_1'(\varkappa) = \varkappa F \geqslant 0$$

Since $\Phi_1(0) = 0$ and $\Phi'_1(\varkappa) \geqslant 0$, it follows that $\Phi_1(\varkappa) \geqslant 0$ (and $\Phi'(\varkappa) \geqslant 0$).

But since $\Phi(0) = 0$ and $\Phi'(\varkappa) \geqslant 0$, it follows that $\Phi(\varkappa) \geqslant 0$.

Appendix 2. To investigate the behavior of trajectories for large $y > 0$ we set $y = 1/\rho$. System (0.1) then becomes the system

$$\frac{d\varphi}{dt} = \frac{1}{\rho}, \qquad \frac{d\rho}{dt} = \rho(\lambda - k\cos\varphi) + \rho^2(\sin\varphi - \gamma)$$

or the equation

$$\frac{d\rho}{d\varphi} = \rho^2(\lambda - k\cos\varphi) + \rho^3(\sin\varphi - \gamma) \tag{A}$$

where ρ and φ can be regarded as ordinary polar coordinates on a plane perpendicular to the axis of the phase cylinder. The substitution $y = 1/\rho$ transforms spirals girding cylinder into spirals girding the equilibrium state at the point $\rho = 0$. The solution of Eq. (A) defined by the initial condition $\rho = \rho_0 > 0$ for $\varphi = 0$ can be sought in the form of the series

$$\rho = \rho_0 u_1(\varphi) + \rho_0^2 u_2(\varphi) + \rho_0^3 u_3(\varphi) + \rho_0^4 u_4(\varphi) + \ldots$$

which converges for all φ in the range $-\pi \leqslant \varphi \leqslant \pi$ and for all sufficiently small values of ρ_0. Determining the functions $u_1(\varphi) \equiv 1$, $u_2(\varphi)$, ... successively from the recursion equations using standard methods, and then setting $\varphi = 2\pi$, we find the succession function on the segment $\varphi = 0$. The equation

$$\rho_1 - \rho_0 = \rho_0^2 u_2(2\pi) + \rho_0^3 u_3(2\pi) + \rho_0^4 u_4(2\pi) + \ldots \equiv \rho_0^2 \{2\pi\lambda + [(2\pi\lambda)^2 - 2\pi\gamma]\rho_0 + [(2\pi\lambda)^3 - 2\pi\lambda(5\pi\gamma + 1) - k\pi]\rho_0^2 + \ldots\} = 0$$

for $\gamma = 0$ and small λ (for any $k > 0$) has a positive root corresponding to the stable limit cycle on the upper phase half-cylinder. For $\gamma = 0$, by virtue of the symmetry of the phase space trajectories (of the strip $-\pi \leqslant \varphi \leqslant \pi$) with respect to the origin, the lower half-cylinder also contains a symmetrically situated limit cycle. Both limit cycles are preserved for small γ. For small γ and λ and any $k > {}^3/_4\,\pi\gamma$ we have the decomposition structure of the phase cylinder shown in Fig. 2(*1*).

Appendix 3. The values of the constant h which isolate the curves of the conservative system near which limit cycles surrounding the equilibrium state exist for small γ and λ are the roots of the equation

$$\psi_3^*(h) = \int_c [\gamma - \lambda(1 - d\cos\varphi)\,y]\,d\varphi = 0$$

where c is one of the curves $y^2 = 2(\cos\varphi + h)$ for $-1 < h < 1$. Since

$$\psi_3^*(h) = -2\sqrt{2}\,\lambda \int_{-\varphi_0}^{\varphi_0} (1 - d\cos\varphi)(\cos\varphi + h)^{1/2} d\varphi \qquad (\cos\varphi_0 + h = 0)$$

so that the roots of the equation $\psi_3^*(h) = 0$ do not depend on γ and λ, it follows that the domain of the plane λd for which there is a limit cycle surrounding the equilibrium state is bounded by the straight line $d = d_1$ (to within terms of order μ if we set $\gamma = \mu\gamma_0$ and $\lambda = \mu\lambda_0$). The value of d_1 can be readily determined in the limiting case $h = 1$ (the closed contour consists of two saddle-to-saddle separatrices). Hence, $d_1 = 3$.

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